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'Hyperspace' (The Cobordism Theory of Space-Time)

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Abstract

It is demonstrated that a compact space and time-orientable space-time is cobordant in the unoriented sense, that is, bounds a compact five-manifold. The bounding property is a direct consequence of the triviality of the Euler number.

Most science-fiction addicts are familiar with the notion of 'Hyperspace' a higher dimensional space-time bounded by Space-Time through which, in the far distant future, intersteller voyagers short-cut the (otherwise unsurmountable) distances between the stars. The purpose of this article is to demonstrate that any compact, but otherwise physically reasonable relativistic space-time model is the boundary of some compact, connected, five-dimensional hyperspace. That is, any space and time-orientable space-time is cobordant in the unoriented sense. The physical interest of cobordism theory lies in attempting to answer the following type of question. Given two compact, disjoint regions X_1 and X_2 of Space-Time X, is there a compact, connected five-manifold Z with $\partial Z = X_1 \cup X_2$? In the following work we prove a simple lemma about compact space and time-orientable space-times and provide a brief review of some of the machinery we shall use in the proof of the main theorem.

Definition 1. A space-time is any paracompact, Hausdorff, smooth, connected four-manifold X admitting a smooth Lorentzian structure.

A Lorentzian structure is a reduction of the Einstein bundle $GL(4)(X)$ of X to the Lorentz group L. The algebraic structure of L allows further refinement of this definition. L has as normal subgroups the proper Lorentz group L_+ of orientation preserving endomorphisms of Minkowski space, the orthochronous Lorentz group $L\uparrow$ of time-sense preserving endomorphisms and the proper orthochronous Lorentz group $L_+\uparrow = L_+ \cap L\uparrow$.

A reduction of $L(X)$ to L_+ is called an orientation of X, a reduction to $L\uparrow$ a time-orientation of X and a reduction to $L_+\uparrow$ a simultaneous orientation and time-orientation of X. If $L(X)$ reduces to $L\uparrow$, L_+ or $L_+\uparrow$, X is called respectively time-orientable, orientable or space and time orientable.

Lemma. A compact space and time orientable space-time has a trivial Euler number.

Proof: Given that the Einstein bundle reduces to $L_+\uparrow$, the Lorentz bundle $L_+\uparrow(X)$ reduces to $SO(3)$. This is because the associated bundle $L_+\uparrow/SO(3)(X)$ has a contractible fibre ($\mathbb{R}^3$) and hence a smooth global section. But such a section is the assignment of a point to each future-oriented timelike hyperboloid in the tangent bundle of X, that is, a global non-vanishing vector field. Thus by the Poincaré–Hopf theorem, $\chi(X) = 0$.

The Euler number $\chi(X)$ of X is only one (but the most important) of the topopological invariants of X that we shall be using in the proof of the main theorem. The following is a brief review of some of the more complicated invariants (Hirzebruch, 1966). In this section, X will stand for a compact, oriented $4k$-manifold with $k \geqslant 1$.

(i) Associated with X are the graded cohomology algebras $H^*(X, \mathbb{Z}_2)$ $H^*(X, Z)$ and $H^*(X, \mathbb{Q})$ where Z is the ring of integers, $\mathbb{Z}_2$ is the field of mod(2) integers and Q is the field of rational numbers. There is a homology class $X^* \in H_{4k}(X, \mathbb{Z}_2)$, the $\mathbb{Z}_2$-orientation of X and a homology class $X_* \in H_{4k}(X, \mathbb{Z})$ called the orientation class of X.

(ii) In each dimension $0 \leqslant i \leqslant 4k$ there are the Stiefel-Whitney characteristic classes $w_i \in H^i(X, \mathbb{Z}_2)$ which, together with the Z_2-orientation class X^* and the Kronecker product, define the Stiefel-Whitney numbers:

$$w_{i_1}^{r_1} \smile \ldots \smile w_{i_p}^{r_p}(X^*) \qquad \text{where } i_1 r_1 + \ldots + i_p r_p = 4k$$

A necessary and sufficient condition for X to be orientable is that $w_1 = 0$. The class w_{4k} is related to the Euler number $\chi(X)$ by $w_{4k}(X^*) = (\chi(X)) \bmod (2)$.

(iii) In dimensions $\equiv 0 \bmod (4)$ there are Pontryagin characteristic classes $P_i \in H^{4i}(X, Z)$ which, together with the Z-orientation class X_* and the Kronecker product, define Pontryagin numbers:

$$P_{i_1}^{r_1} \smile \ldots \smile P_{i_p}^{r_p}(X_*) \qquad \text{where } i_1 r_1 + \ldots + i_p r_p = k$$

The Pontryagin classes are related to the Stiefel–Whitney classes by $w_{2i}^2 \equiv P_i \bmod (2)$.

(iv) Another topological invariant of compact oriented $4k$-manifolds is the signature S_X defined as follows. The cup-product and the **Z**-orientation class define a bilinear, symmetric, non-degenerate form b_X on $H^{2k}(X, \mathbb{Q})$ by

$$H^{2k}(X, \mathbb{Q}) \, \mathbb{Q} \, H^{2k}(X, \mathbb{Q}) \xrightarrow{\smile} H^{4k}(X, \mathbb{Q}) \cong \mathbb{Q}$$

The signature of X is defined as the signature of b_X, i.e.: $p - n$ where p is the number of positive and n is the number of negative entries in the diagonalised form of b_X. The signature and the Euler number are related by $S_X \equiv \chi(X)$ mod(2).

Definition 2. Two compact manifolds X_1 and X_2 are cobordant iff $X_1 \cup X_2 = \partial Z$ for Z a compact manifold.

The set M_0 of cobordism classes of compact manifolds is a graded Z_2-algebra under the operations induced by the product and sum of manifolds. The structure of the cobordism algebra was determined by Thom (1954) who showed that two compact manifolds are cobordant iff they have the same Stiefel–Whitney numbers. The class of the empty manifold forms the additive identity of the cobordism algebra and a manifold is called cobordant iff it is cobordant to the empty manifold, that is, iff it bounds a compact manifold. Thus in order to show that a compact manifold is a boundary, it is sufficient to show that it has trivial Stiefel–Whitney numbers.

Theorem. Any compact space and time-orientable space-time is cobordant.

Proof. We show that all the Stiefel–Whitney numbers of a compact space and time-orientable space-time are trivial. By lemma 1 a compact space and time orientable space-time has trivial Euler number. Hence because $w_4(X^*) = [\chi(X)] \bmod(2)$, the top Stiefel–Whitney number is trivial. Also because X is orientable, $w_1 = 0$ which implies that $w_3 . w_1(X^*)$, $w_2 . w_1{}^2(X^*)$ and $w_1{}^4(X^*)$ are trivial. The only remaining number is $w_2{}^2(X^*)$. To show that this also vanishes we make use of the Hirzebruch Signature Theorem $P_1(X_*) = 3 . S_X$ (Hirzebruch, 1966). From this we obtain $P_1(X_*) \equiv S_X \bmod(2)$ and hence using $w_2{}^2(X^*) = P_1(X_*) \bmod(2)$ and $\chi(X) \equiv S_X \bmod(2)$, we obtain $w_2{}^2(X_*) = \chi(X) \bmod(2)$. Hence $\chi(X) = 0$ implies that $w_2{}^2(X) = 0$.

One can define a cobordism relation in the category of oriented manifolds.

Definition 3. Two compact, oriented manifolds X_1^+ and X_2^+ are cobordant as oriented manifolds iff $X_1^+ \cup X_2^- = \partial_0 Z$, the oriented boundary of a compact oriented manifold Z.

The set M_{SO} of cobordism classes is a graded ring with the operations induced by the sum and product of manifolds. Its structure was determined by Wall (1960) who also showed that two compact oriented manifolds are cobordant as oriented manifolds iff they have the same Pontryagin and Stiefel–Whitney numbers. Thus a compact oriented manifold is an oriented boundary iff its Pontryagin and Stiefel–Whitney numbers are all trivial. Specialising to compact space and time-orientable space-times, the only obstruction to choosing an *oriented* manifold bounded by the space-time is the Pontryagin number $P_1(X_*)$ which we know to be even. Because $P_1(X_*) = 3S_X$, the obstruction is involved with the group $H^2(X, \mathbb{Q})$, in particular, if for example, $H^2(X, \mathbb{Q}) = 0$, X is an oriented boundary.

Another type of cobordism relation was defined by Reinhart (1962), vector cobordism.

Definition 4. Two compact manifolds X_1 and X_2 are vector-cobordant iff they are cobordant in the unoriented sense and if they together bound a compact manifold with a non-zero vector field interior oriented on X_1 and exterior oriented on X_2.

The Euler number turns out to be a vector cobordism invariant and Reinhart showed that two compact manifolds are vector-cobordant iff they have the same Euler and Stiefel–Whitney numbers. Therefore because a compact space and time-orientable space-time has zero Euler and Stiefel–Whitney numbers it is vector cobordant. That is, any compact space and time orientable space-time is the boundary of a compact connected manifold carrying a non-zero vector field normal to an interior oriented on the bounding space-time. In our 'hyperspace' interpretation of cobordism, not only is any reasonable compact space-time the boundary of some five-dimensional hypermanifold, but there is an everywhere well-defined direction in the hypermanifold, an extra 'time dimension'.

Another application of vector-cobordism has been discussed by Reinhart (1962), Geroch (1967) and Yodzis (1972, 1973). Given a time-orientation of a time-orientable space-time (a non-zero timelike vector field) two compact hypersurfaces transverse to the time direction which together bound a compact region of the space-time are vector cobordant. Does this mean that the two manifolds need be topologically related in a sense more narrow than vector-cobordism? To answer such a question one looks at the vector-cobordism group M_V^3. Because the latter is trivial, there need not be any relationship. Reinhart (1964) also defined codimension-one foliated cobordism. In this theory, the vector field occurring in the definition of vector cobordism has to be transverse to a smooth codimension-one foliation of the bounded manifold. As yet, the structure of the foliated cobordism groups is unknown. They have the following applications to space-time topology. Suppose we ask for the time-orientation vector field of a time-orientable space-time to be transverse to a foliation of the space-time into space-like hypersurfaces. Then any two compact leaves which together bound a compact subspace of the space-time are cobordant in the foliated sense. Motivated by results like the Reeb stability theorem, one would probably obtain some non-trivial relationship between the hypersurfaces.

Returning to hyperspace, we have shown that any two compact, space and time-orientable space-times are cobordant in the unoriented sense, and which is a stronger result, they are vector cobordant. Thus if X_1 and X_2 are any two compact, space and time-orientable space-times, there is a compact, connected five-manifold Z with a vector field vanishing nowhere, interior normal on X_1 and exterior normal on X_2. Call this the 'transfer' vector field (a sort of natural navigational aid in hyperspace!). If we ask for the transfer field to be transverse to a smooth codimension-one foliation of hyperspace, our hypothetical space traveller is always in some one of the ordered family of 'space-times' between X_1 and X_2. To him, the character of space-time would seem to metamorphise into his desired space-time destination.

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